Lecture VII

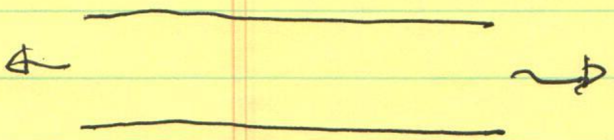
Winter 2017

Aside: Convection in "tall, thin" box (Question from LC).

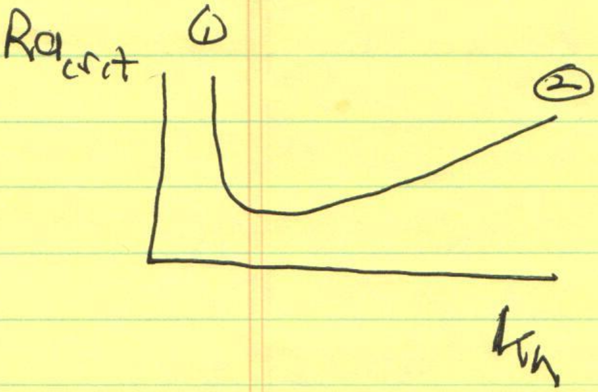
Recall:

- 'lesson' from study of Rayleigh–Benard convection is that boundary conditions matter!

- recall case study was "short, wide" box



- dismissed side
- focus on stress-free / no slip top, bottom.



② →

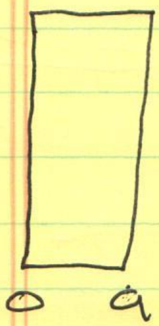
increasing diffusive damping due high k_h

①

increasing diffusive damping due small vertical scales ($\sim$ high k_z).

tall, thin box

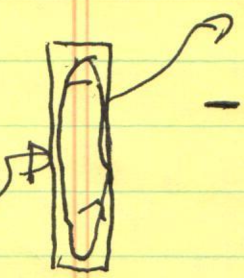
Now:



→ ignore top/bttm boundary

→ no-slip on side walls is key → and only relevant case

i.e. $V_z \Rightarrow w$; $w(0) = w(a) = 0$



- long thin cell must fight no-slip b.c.'s on side wall

- higher k_h will introduce high k_y damping.

∴ - expect high $Ra)_{crit}$ due side-wall no slip, even at $k_h \to k_{h\,min}$.

- curvature of Ra_{crit} vs k_h curve TBD. Speculate rather weak curvature.

- as side surface area >> top surface area, expect top no-slip vs stress free comparison not significant.

(a) → Introduction to Rotating Convection
— Intermezzo on Landau Theory.

(b) → Basics of Waves.

A.) Convection + Rotation

- see Lecture 6 for $\begin{cases} \text{Rotation} \\ \text{Freezing-in Law} \\ \text{Taylor-Proudman Thm.} \end{cases}$

Key pt: for Ω large enough, flow is two-dimensionalized

→ Inertial Waves
— Rotating Convection

→ Inertial Waves → (radial) buoyancy waves in rotating fluid

~> Recall Pblm 4, Set 1:

Fluid rotates at $\Omega \hat{z}$, $k_\theta = 0$ (convenience), $k_r, k_z \neq 0$

$$\Rightarrow \omega^2 = k_z^2 (4\Omega^2) / k_z^2 + k_r^2$$

and, with radial b.c., eigenvalue.

→ Quick derivation:

- from vorticity equation (T-P thm.), linearization:

$$\frac{\partial \tilde{\underline{\omega}}}{\partial t} + \cancel{\underline{v}\cdot\underline{\nabla}(\underline{\omega}+2\underline{\Omega})} = 2\Omega\frac{\partial}{\partial z}\tilde{\underline{v}} + \cancel{\tilde{\underline{\omega}}\cdot\underline{\nabla}\,\tilde{\underline{v}}}$$

so

$$\partial_t\,\tilde{\omega}_z = 2\Omega\,\partial_z\tilde{v}_z$$

and, from EOM

$$\frac{\partial \tilde{\underline{v}}}{\partial t} + \cancel{\tilde{\underline{v}}\cdot\underline{\nabla}\,\tilde{\underline{v}}} = -\frac{\underline{\nabla}}{\rho}p^{+} + \underline{v}\times 2\Omega\hat{z}$$

$\underline{\nabla}\times\underline{\nabla}\times\underline{v}$ ⇒

$$\partial_t\,\underline{\nabla}(\underline{\nabla}\cdot\tilde{\underline{v}}) - \partial_t\nabla^2\tilde{\underline{v}} = 0 + \underline{\nabla}\times\underline{\nabla}\times\left[\underline{v}\times 2\Omega\hat{z}\right]$$

so

$$-\partial_t\nabla^2\tilde{\underline{v}} = \underline{\nabla}\times\left(2\Omega\partial_z\tilde{\underline{v}} - \cancel{\underline{\nabla}\cdot\underline{v}\,2\Omega\hat{z}}\right)$$

so, Σ.

$$-\partial_t \nabla^2 \tilde{V}_z = 2\Omega\, \partial_z \tilde{\omega}_z$$

axial gradient in $\tilde{\omega}$ ⇒ axial acceleration

and:

$$\partial_t \tilde{\omega}_z = 2\Omega\, \partial_z \tilde{V}_z$$

stretching vortex ⇒ vorticity fluctuation.

⇒ inertial wave dispersion relation

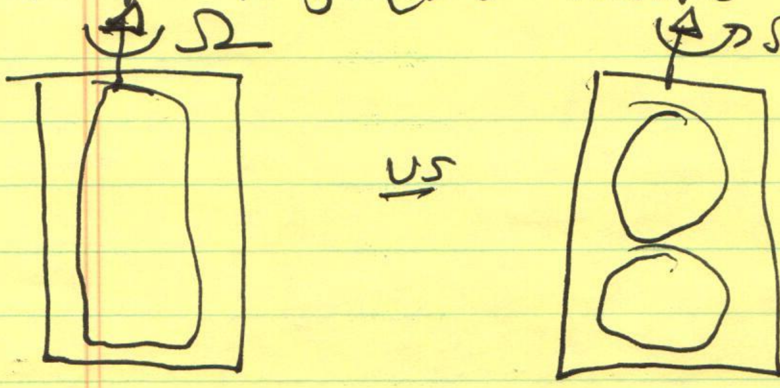
$$\omega^2 = k_z^2\, 4\Omega^2 / (k_r^2 + k_z^2)$$

- physics is rotating flow vortex lines don't like being 'bent' ($k_z \neq 0$)
  ⇒ imposes energy penalty for motions with finite k_z
  ⇒ ⊕ definite contribution to δW.
- rough picture is one of gyroscopic restoring force (conservation L_z)

- analogous to Alfven wave and field line bending in MHD.
  $\omega^2 = k_\parallel^2 V_A^2$

- aside: - backward wave: $V_{gr} < 0$.
  - $\omega = 0$ finite k_r modes.
    (shear layers).

⇒ as convection necessitates cellular overturning (i.e. finite k_z),



low k_z ($k_{z\,\min}$) motions favored

⇔ tracks T-P thm. conclusion of 2D-ization of the flow.

→ suggests that rotation is (strong) stabilizing effect on convection.

Also suggests that another dimensionless number enters comparing rotation to viscous dissipation
⇒ naturally:

$$Ta = 4\Omega^2 d^4/\nu^2 \sim \frac{(2\Omega)^2}{(\nu/d^2)^2} \qquad d \equiv \text{box scale}$$

Taylor Number

~ Taylor number captures natural competition between rotation and viscous diffusion.

~ Ta joins Ra, Pr as key parameter in convective stability theory

~ $Ra_{crit} = Ra_{crit}(Pr, Ta)$ is now stability threshold problem.

N.B. Ra, Ta both involve ν but are distinct — $\alpha g \Delta T/d$ vs Ω^2.

Can combine stationary convection and inertial wave calculations to obtain basic equations:

$$\partial_t \Theta = \Theta W + \kappa \nabla^2 \Theta$$

$$\partial_t \nabla^2 W = g\alpha \nabla_h^2 \Theta + \nu \nabla^2 \nabla^2 W - 2\Omega\, \partial_z \omega_z$$

$$\partial_t \omega_z = -2\Omega\, \partial_z W$$

(derive)

notation as before.

For ideal stability:

$$\boxed{\omega^2 = \left[-g\alpha\beta k_h^2 + (4\Omega^2) k_z^2\right] / k^2}$$

- favor high k_h, low k_z cells

$\Rightarrow$

- Taylor columns (thin) and Proudman pillars, as shown in movies.

- stabilizing effect of rotation evident

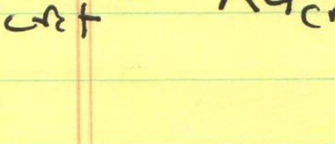
For viscous, conductive stability with rotation, then:

$$Ra_{crit} = Ra_{crit}(Ta, \alpha \text{[redacted]}) , \quad \text{for } P_r \sim 1 .$$

$$\Downarrow$$

$$k_h h$$

One can further specify $Ra_{,\alpha}$ as Ra_{crit} minimum (vs $\alpha = k_h h$)

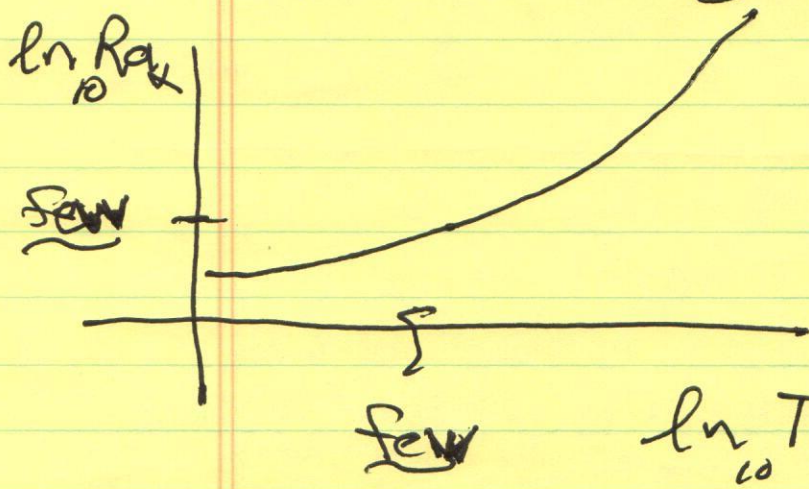


$$Ra_\alpha = Ra_\alpha(Ta)$$

↓

Taylor # dependence

For all cases: { no-slip / stress free }



{ demonstrates stabilizing effect rotation.

→ can develop variational principle for exchange-of-stabilities case. Need also treat over-stable limit.

→ Cultural Aside: Magnetoconvection

Dynamo-generated magnetic fields can feed back on convection. A particular example is sunspots (i.e. dark - lower T ?) - convection weakened), which are

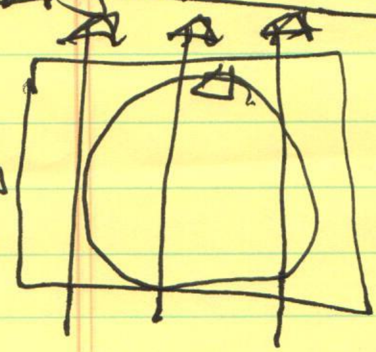
N.B.: Comment on dissipation effects.

associated with strong magnetic fields

⇒ Magnetoconvection

B_0

$\nabla T < 0$



δg

- similar to rotation problem, with 'bending element' due B_0
  ⇒ i.e. energy penalty
- Alfven wave replaces inertial wave.
- expulsion can occur, etc.

Enough linear stability theory!

Intermezzo

Commentary: Landau Equation / Law.

← to date, linear stability

→ Nonlinear evolution ??

⇒ difficult problem, especially for turbulence...

⇒ seek characterize weakly nonlinear evolution, i.e. for flow shear / KH instability (stabilized by viscosity) then $\exists$ Re_{crit} $\rightarrow$ critical Reynolds number for instability. Equivalently for convection, Ra_{crit} exists. Then, if

$$Re \cong Re_{crit} + \delta Re \qquad \qquad \frac{\delta Re_a}{Re_{a\,crit}} \ll 1$$

$$Ra \cong Ra_{crit} + \delta Ra$$

can one represent dynamics, in same general form? especially near marginal point?

Analogy: $\rightarrow$ Ginzburg-Landau theory

Leverage: $\rightarrow$ Symmetry!

So, if consider N-S E retaining nonlinear terms: $\underline{V} = \underbrace{\underline{V}_0}_{\text{mean}} + \tilde{\underline{V}}_1$ $\rightarrow$ perturbation

$$\frac{\partial}{\partial t}\tilde{V} + \tilde{\underline{V}}\cdot\underline{\nabla}\,\underline{V}_0 + \underline{V}_0\cdot\underline{\nabla}\tilde{V}_1 + \underbrace{\tilde{\underline{V}}_1\cdot\underline{\nabla}\,\tilde{\underline{V}}_1}_{\text{NL}} = -\frac{\nabla\tilde{p}_1}{\rho} + \nu\nabla^2\tilde{\underline{V}}_1 .$$

- For $\delta Re \ll Re_{crit}$ might expect few/one mode relevant just above marginality;

and

in general, ω_r develops, oscillations occur. e.g. KH.

$$\tilde{\underline{V}}_1 = \underline{f}_1(\underline{r})\, e^{(\gamma_1 t - i\omega_1 t)}$$

often:

$$= \underbrace{\underline{f}_1(\underline{r})}_{\text{envelope}}\, \overbrace{e^{i\underline{k}_1\cdot\underline{r}}}^{\text{spatial carrier}}\, e^{-i\omega_1 t}\, e^{\gamma_1 t}$$

then for convenience:

$$\tilde{v}_1 = \underbrace{A(t)}_{\text{amplitude}}\, \underline{f}_1(r)$$

so

$$\frac{d}{dt}|A|^2 = \overbrace{2\gamma_1 |A|^2}^{\text{linear growth.}} + O(\underbrace{A^3}_{|A|^2 A}) + O(A^4)$$

etc.

Now: for $\gamma \sim Re - Re_{crit}$

$$\omega_r \to \text{finite}$$

i.e. time scale separation between growth and oscillation, i.e. $\omega_r > \gamma$

then $\overline{A^2} = \int_0^{2\pi/\omega_r = \tau} \frac{dt}{\tau}$

↳ period avg.

Now: $\overline{\tilde{\underline{V}}_1 \cdot (\tilde{\underline{V}}_1 \cdot \underline{\nabla}\, \tilde{\underline{V}}_1)} \to 0$ single mode (multi-mode ⟷ resonant coupling).

so $O(A^3)$ contribution vanishes.

Now, $O(A^4)$, i.e.

$$\overline{\tilde{\underline{V}}_1 \cdot [\tilde{\underline{V}}^{(2)} \cdot \underline{\nabla}\, \tilde{\underline{V}}_1]} \;?$$

but: $\tilde{V}^{(2)} \sim \tilde{V}_1 \tilde{V}_1$

$$\Rightarrow \overline{O(|A|^4)} = \overline{\tilde{\underline{V}}_1 \cdot \tilde{\underline{V}}_1 \tilde{\underline{V}}_1 \cdot \nabla \tilde{V}_1}$$

$$\sim -2\,(|A|)^4$$

$$\boxed{\partial_t |A|^2 = 2\gamma_1 |A|^2 - \alpha |A|^4}$$

→ Landau equation → obvious structural / conceptual similarity to Ginzburg-Landau theory.

→ physics is mode feed-back on profile, i.e.

$\underline{\tilde{V}}_1 \cdot \underline{\tilde{V}}_1 \cdot \nabla \underline{\tilde{V}}^{(2)}$ as:

$$\overline{\underline{\tilde{V}} \cdot \underline{\tilde{V}} \cdot \underline{\nabla}(\underline{\tilde{V}})} = \underline{\tilde{V}} \cdot \underline{\tilde{V}} \cdot \underline{\nabla}\overline{\left(\underset{\substack{\downarrow \\ \text{mean profile} \\ \text{(driving shear)}}}{\overline{V_0}} + T(\underline{\tilde{V}}_1, \underline{\tilde{V}}_1)\right)}$$

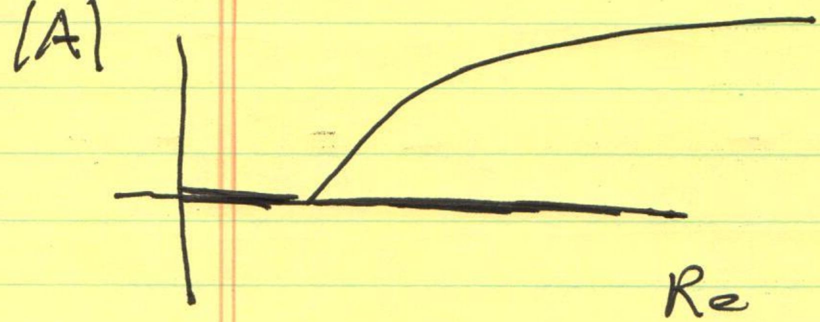
$\rightarrow |A|^4$

nonlinearity acts to deplete free energy source.
modify

→ predicts saturation at:

$$|A|^2 \simeq 2\gamma_1 / \alpha \simeq (Re - Re_{crit})$$

so $|A| \sim (Re - Re_{crit})^{1/2}$ } stationary state



super-critical bifurcation

→ calculation of α in detail via reductive perturbation theory

⇒ come back for 221a in spring → extensive discussion of weakly nonlinear convection rolls

but:

- α need not be positive. In this case $O(|A|^4)$ contribution → growth/destabilization. So:

  → need $O(|A|^6)$ to saturate

  → strong enough perturbations can grow, even if linearly stable

In this case: Landau Eqn. becomes,

$$\partial_t |A|^2 = 2\gamma_i |A|^2 + |\alpha| |A|^4 - \beta |A|^6$$

$$\Rightarrow -2|\gamma_i| |A|^2 + |\alpha| |A|^4 - \beta |A|^6$$

so, for stationary state:

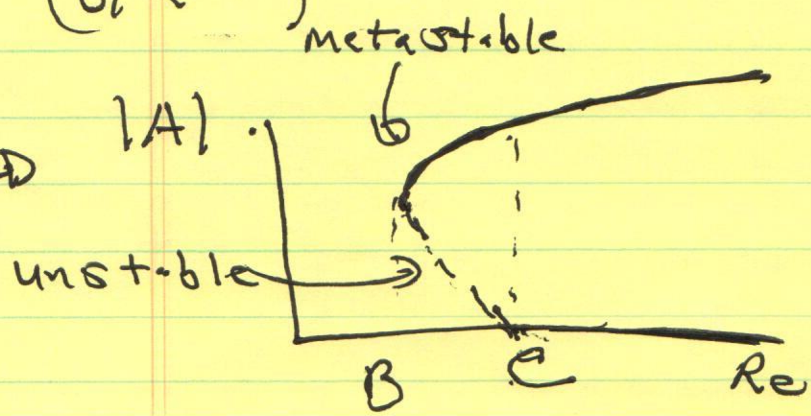
$$|A|^2 \left[-2|\gamma_i| + |\alpha| |A|^2 - \beta |A|^4 \right] = 0$$

$$|A|^2 = + \frac{|\alpha|}{2\beta} \pm \left(\frac{|\alpha|^2}{4\beta^2} - 4 \frac{(2|\gamma_i|)}{\beta} \right)^{1/2}$$

$|A|^2 = 0$ (2 roots)

→ subcritical (nonlinear) instability / bifurcation possible if $|A|^2 > \frac{2|\gamma_i|}{|\alpha|}$

($\gamma_i < 0$)

→ 

$\alpha < 0$

$|A| \Leftrightarrow \eta$ order parameter

supercritical bifurcation $\Leftrightarrow$ 2^{nd} order transition

subcritical bifurcation $\Leftrightarrow$ 1^{st} order transition

(both exhibit metastable state)

$\gamma_L(Re) \Leftrightarrow a(T-T_c)$ factor

— can also develop Landau theory for phase and amplitude

i.e. $\underline{V} \rightarrow A e^{i\phi}$

$\Rightarrow$ phase dynamics

$\rightarrow$ CGL system, etc.